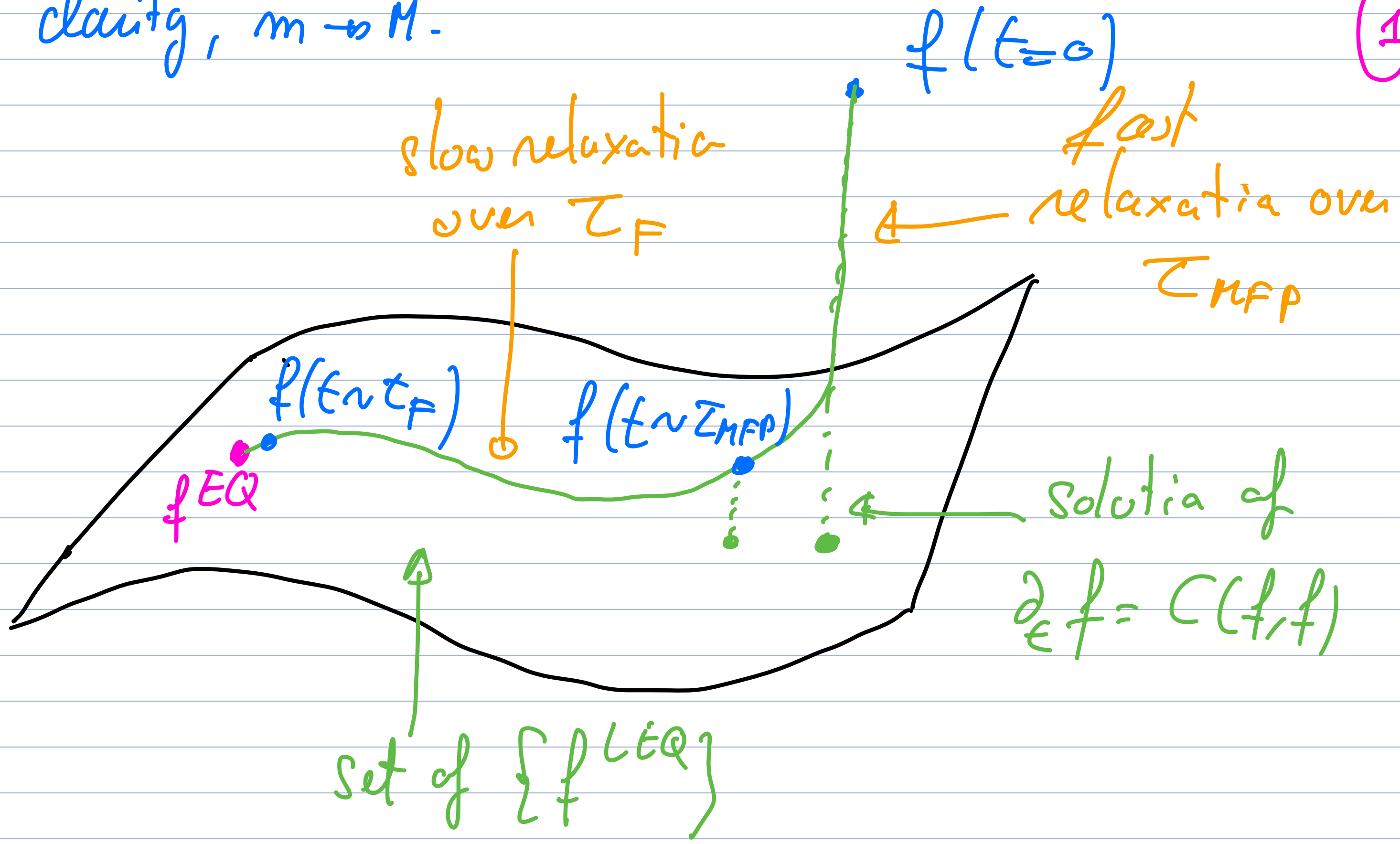
For clarity, $m \to M$.



BE: $\partial_t f + \underbrace{\frac{\partial f}{\partial \vec{q}} \cdot \frac{\vec{p}}{M}}_{\equiv \frac{1}{\tau_F} L_F f} = \underbrace{C(f,f)}_{\equiv \frac{1}{\tau_{MFP}} \hat{C}(f,f)}$; $t = \hat{t}\,\tau_F$; $\varepsilon = \frac{\tau_{MFP}}{\tau_F}$

$$(BE)\times\tau_F: \quad \partial_{\hat{t}} f + L_F f = \frac{1}{\varepsilon}\hat{C}(f,f) \qquad (*)$$

$$f(\vec{q},\vec{p},t) = f_0(\vec{q},\vec{p},t) + \varepsilon f_1(\vec{q},\vec{p},t) + O(\varepsilon^2)$$

Order by order:

$O(\frac{1}{\varepsilon})$ $\quad 0 = C(f_0,f_0) \Rightarrow f_0 = f^{LEQ}$

$O(1)$ $\quad \partial_{\hat{t}} f_0 + L_1 f_0 = \hat{C}(f_0,f_1) + \hat{C}(f_1,f_0) \Rightarrow$ dynamics of f_0 due to transport, $L_F f_0$ & collisions, $\hat{C}(\dots)$

$O(\varepsilon)$ $\quad \partial_{\hat{t}} f_1 + \dots \Rightarrow$ dynamics of f_1

Leading order

$$f_0(\vec{q},\vec{p},t) = f^{LEQ}(\vec{p},\vec{q}) = \tilde{\gamma}(q,t)\, e^{-\vec{\alpha}(\vec{q},t)\cdot\vec{p} - \beta(\vec{q},t)\left[\frac{\vec{p}^{\,2}}{2m} + U(\vec{q})\right]}$$

Q: Can we understand $\tilde{\gamma}$, $\vec{\alpha}$ & β?

f_0 is a Gaussian function of $\vec{p} \Rightarrow$ characterized by its moments & normalization

Slow fields:

density $n(\vec{q},t) = \int d\vec{p}\, f(\vec{q},\vec{p},t) \Rightarrow$ normalization of f

momentum field $\vec{w}(\vec{r},t) \equiv \int d\vec{p}\, \vec{p}\, f \Rightarrow 1^{st}$ moment of f

kinetic energy field $K(\vec{r},t) = \int d\vec{p}\, \frac{\vec{p}^{\,2}}{2m} f \Rightarrow 2^{nd}$ moment of f

$\Rightarrow$ we can reparametrize f in terms of the conserved fields

As anticipated, the knowledge of the slow field allow to determine all other coarse-grained field, and in particular f.

First order: single time approximation

* $\partial_t f = C(f,f)$ makes f relax to f_0 in a time $t \sim O(\tau_{MFP})$

$$\Rightarrow C(f,f) = -\frac{1}{\tau_{MFP}}(f-f_0) = -\frac{1}{\tau_{MFP}}\cdot \varepsilon f_1 = -\frac{1}{\tau_F} f_1$$

$\Rightarrow$ makes f_1 relax in a time τ_F to zero, as anticipated.

Rescaled dynamics: $\frac{1}{\varepsilon}\hat{C} = \frac{1}{\varepsilon}\tau_{MFP} C = -\frac{\varepsilon}{\varepsilon} f_1$ makes f_1 relax to 0

in a time $\hat{t} \sim O(1)$, i.e. $t \sim O(\tau_F) \Rightarrow$ consistent

$\Rightarrow$ $O(1)$: $\partial_{\hat{t}} f_0 + L_1 f_0 = -f_1$

$$\Rightarrow f_1 = -[\partial_{\hat{t}} f_0 + L_1 f_0] = -\tau_F [\partial_t f_0 + \{f_0, H_1\}]$$

× If we know $\partial_t f_0$ to order 0 in ε, then we know f_1.

× $\partial_t f_0$ is related to $\partial_t n$, $\partial_t \vec{u}$ & $\partial_t k$, the evolution of the slow modes, which are our ultimate goal $\Rightarrow$ let's derive them!

2.4.2) The hydrodynamic equations

Evolution of the density field

$n(\vec{q}) = \int d^3\vec{p}\, f(\vec{q},\vec{p})$

$\int d^3\vec{p}$ (BE) gives

$$\partial_t n + \frac{\partial}{\partial \vec{q}} \cdot \int d^3\vec{p}\, f \frac{\vec{p}}{M} = \int d^3p\, d^3\vec{p}_2\, d^2\sigma\, |\vec{v} - \vec{v}_2| (f' f_2' - f f_2) \times 1$$

$1 \leftrightarrow 2$ & $()' \leftrightarrow ()$

$$0 = \frac{1}{4} \int d^3p\, d^3\vec{p}_2\, d^2\sigma\, |\vec{v} - \vec{v}_2| (f' f_2' - f f_2) \times \underbrace{(1+1-1-1)}_{=0}$$

as for the H theorem

$\Rightarrow \partial_t n = -\frac{\partial}{\partial \vec{q}} \cdot \vec{j}(\vec{q},t) \; ; \; \vec{j} = \int d^3\vec{p}\, f\, \vec{v}$

Comment: f is locally proportional to the number of particles $\Rightarrow$ so is $\vec{j}$.

$\Rightarrow$ This suggests defining a local velocity field through

$$\vec{j}(\vec{q},t) \equiv n(\vec{q},t)\, \vec{u}(\vec{q},t) \, , \text{ where}$$

$$\vec{u}(\vec{q},t) \equiv \frac{1}{n(\vec{q},t)} \int d^3\vec{p}\, f(\vec{q},\vec{p},t)\, \vec{v} = \frac{\int d^3\vec{p}\, \vec{v}\, f(\vec{q},\vec{p},t)}{\int d^3\vec{p}\, f(\vec{q},\vec{p},t)}$$

Note that $g_{f,\vec{q}}(\vec{p}) = \dfrac{f(\vec{q},\vec{p},t)}{\int d^3\vec{p}\, f(\vec{q},\vec{p},t)}$ is a normalized probability distribution for $\vec{p}$ "at $\vec{q}$". If we denote by $\langle \ldots \rangle_f$ the corresponding average, we have $\vec{u}(\vec{q},t) = \langle \vec{v} \rangle_f$

Evolution for the density field $\boxed{\partial_t n(\vec{q},t) = -\frac{\partial}{\partial \vec{q}} \cdot \left(\vec{u}(\vec{q},t)\, n(\vec{q},t)\right)}$ (1)

Material derivative (1) $\Leftrightarrow$ $\partial_t n + \vec{u} \cdot \frac{\partial}{\partial \vec{q}} n = -n \frac{\partial}{\partial \vec{q}} \cdot \vec{u}$

Let us define $D_t \equiv \partial_t + \vec{u} \cdot \frac{\partial}{\partial \vec{q}} = \partial_t + u_\alpha \partial_{q_\alpha}$, where $\partial_{q_\alpha} = \frac{\partial}{\partial q_\alpha}$

5

and summations over repeated indices are implicit. The dynamics of $n(\vec{q},t)$ then becomes

$$\boxed{D_t n = -n\,\partial_{q_\alpha} u_\alpha} \qquad (1')$$

$\Rightarrow$ Need to characterize $\vec{u}(q,t)$.

Evolution of the velocity field

$$u_\alpha = \frac{1}{n}\int dp\, v_\alpha f$$

$$\partial_t u_\alpha = -\frac{\partial_t n}{n^2}\underbrace{\int dp\, v_\alpha f}_{n u_\alpha} + \frac{1}{n}\int dp\, v_\alpha \partial_t f$$

$$= \frac{\partial_{q_\beta}(u_\beta n)}{n}\times u_\alpha + \frac{1}{n}\int dp\, v_\alpha\left[-\partial_{q_\beta} f\cdot v_\beta\right] + \frac{1}{n}\int dp\,dp_2\,d\sigma\,|\vec{v}-\vec{v}_2|\,(f'f_2'-ff_2)\,v_\alpha$$

$$= u_\alpha\partial_{q_\beta}u_\beta + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta}n - \frac{1}{n}\partial_{q_\beta}\int dp\, f v_\alpha v_\beta + \frac{1}{4n}\int dp\,dp_2\,d\sigma\,|\vec{v}-\vec{v}_2|\,(f'f_2'-ff_2)$$
$$\times\underbrace{(v_\alpha+v_{\alpha,2}-v'_\alpha-v'_{\alpha,2})}_{=0 \text{ since } P_{CM}=P'_{CM}}$$

$$D_t u_\alpha = \partial_t u_\alpha + u_\beta\partial_{q_\beta}u_\alpha = \underbrace{\partial_{q_\beta}(u_\alpha u_\beta) + \frac{u_\alpha u_\beta}{n}\partial_{q_\beta}n}_{\frac{1}{n}\partial_{q_\beta}(n u_\alpha u_\beta)} - \frac{1}{n}\partial_{q_\beta}\left[n\langle v_\alpha v_\beta\rangle\right]$$

$$D_t u_\alpha = -\frac{1}{n}\partial_{q_\beta}\left[n\left(\langle v_\alpha v_\beta\rangle - \langle v_\alpha\rangle\langle v_\beta\rangle\right)\right] = -\frac{1}{n}\partial_{q_\beta}\left[n\langle v_\alpha v_\beta\rangle_c\right]$$

This suggest introducing the pressure tensor

$$P_{\alpha\beta} = \underset{\substack{\text{particle}\\\text{mass}}}{M} n \left\langle (u_\alpha - v_\alpha)(u_\beta - v_\beta)\right\rangle = M n \langle \delta v_\alpha \, \delta v_\beta\rangle$$

with $\delta\vec{v} = \vec{v} - \langle\vec{v}\rangle = \vec{v} - \vec{u}$

to rewrite the dynamics as

$$\boxed{M D_t u_\alpha = -\frac{1}{n}\partial_{q_\alpha}\cdot P_{\alpha\beta} \Leftrightarrow M D_t \vec{u} = -\frac{1}{n}\vec{\nabla}\cdot\bar{\bar{P}}} \quad (2)$$

Comments: (1) & (2) are like the Navier Stokes equations for a gas.

Evolution of kinetic energy: $\varepsilon = \left\langle \frac{1}{2} m \,\delta\vec{v}^2\right\rangle = \frac{1}{2} m\left(\langle\vec{v}^2\rangle - \vec{u}^2\right)$

Painful algebra leads to

$$\boxed{\partial_t \varepsilon + u_\alpha \partial_\alpha \varepsilon = -\frac{1}{n}\partial_\alpha h_\alpha - \frac{1}{n} P_{\alpha\beta} u_{\alpha\beta}} \quad (3)$$

where

- $u_{\alpha\beta} = \frac{1}{2}\left(\partial_{q_\alpha} u_\beta + \partial_{q_\beta} u_\alpha\right)$ is called the strain rate tensor
- $h_\alpha = \frac{nM}{2}\langle \delta v_\alpha \, \delta v_\beta \delta v_\beta\rangle$ is the kinetic energy flux along $\vec{u}_\alpha$, a.k.a. heat flux

Proof of Eq (3)

$$\partial_t (n\varepsilon) = \partial_t \frac{1}{2} m \left[\int d\vec{p}\; f\,(\vec{v}-\vec{u})^2\right] = \frac{1}{2} m\left[\int d\vec{p}\,(\vec{v}-\vec{u})^2\,\partial_t f\right] + m\left[\int d\vec{p}\; f\,\partial_t \vec{u}\cdot(\vec{u}-\vec{v})\right]$$

depends on $\vec{q}$

$$= -\frac{1}{2} m \int d\vec{p}\,(\vec{v}-\vec{u})^2\,\vec{v}\cdot\frac{\partial f}{\partial \vec{q}} \quad + m\left[\partial_t \vec{u}\cdot n \underbrace{\langle \delta\vec{v}\rangle}_{=0}\right]$$

$$= -\frac{1}{2} m\,\partial_{q_\alpha}\int d\vec{p}\;\delta v_\beta\,\delta v_\beta\, v_\alpha f + \frac{1}{2} m \int d\vec{p}\; f\,\vec{v}\cdot\underbrace{\frac{\partial}{\partial \vec{q}}\cdot(\vec{v}-\vec{u})^2}_{2(\vec{u}-\vec{v})_\beta\,\frac{\partial}{\partial \vec{q}} u_\beta}$$

$$= -\partial_{q_\alpha}\left[\frac{n m}{2}\langle \delta v_\beta\,\delta v_\beta\, v_\alpha\rangle\right] + m(\partial_{q_\alpha} u_\beta)\cdot\underbrace{\int d\vec{p}\; f\, v_\alpha\,(u_\beta - v_\beta)}_{\int d\vec{p}\, f\,(v_\alpha-u_\alpha)(u_\beta-v_\beta) + u_\alpha \underbrace{\int d\vec{p}\, f\,\delta v_\beta}_{=0}}$$

$$= -\partial_{q_\alpha}\left[\frac{n m}{2}\langle \delta\vec{v}^2\, v_\alpha\rangle\right] - (\partial_{q_\alpha} u_\beta)\,P_{\alpha\beta} \qquad (1)$$

$$\partial_t(n\varepsilon) = n\,\partial_t\varepsilon + \varepsilon\,\partial_t n = n\,\partial_t\varepsilon + \varepsilon\left[-\partial_{q_\alpha}(n u_\alpha)\right]$$

$$= n\,\partial_t\varepsilon - \partial_{q_\alpha}\left[n u_\alpha \varepsilon\right] + n u_\alpha\,\partial_{q_\alpha}\varepsilon \qquad (2)$$

$$(2)=(1) \iff n\,\partial_t\varepsilon + n u_\alpha\,\partial_{q_\alpha}\varepsilon = -\partial_{q_\alpha}\left[\frac{n m}{2}\langle \delta\vec{v}^2\, v_\alpha\rangle - n u_\alpha \frac{m}{2}\langle \delta v^2\rangle\right] - P_{\alpha\beta}\,\partial_{q_\alpha} u_\beta$$

$$= -\partial_{q_\alpha}\left[\frac{n m}{2}\langle \delta\vec{v}^2\,\delta v_\alpha\rangle\right] - P_{\alpha\beta}\,u_{\alpha\beta}$$

where $u_{\alpha\beta} = \partial_{q_\alpha} u_\beta + \partial_{q_\beta} u_\alpha$ and we have used $P_{\alpha\beta} = P_{\beta\alpha}$.

All in all

$$\boxed{\partial_t\varepsilon + u_\alpha\,\partial_{q_\alpha}\varepsilon = -\frac{1}{n}\,\partial_{q_\alpha} h_\alpha - \frac{1}{n} P_{\alpha\beta}\,u_{\alpha\beta}}$$

$$\underbrace{h_\alpha = \frac{n m}{2}\langle \delta v_\alpha\,\delta v_\alpha\,\delta v_\beta\rangle}_{\text{kinetic energy flux along } \vec{u}_\alpha}$$

$\Rightarrow$ heat flux

Temperature field

It is useful to define $T(\vec{q},t) = \frac{2}{3}\frac{\varepsilon(\vec{q},t)}{k_B}$ such that $\varepsilon = \frac{3}{2}k_B T$ and

$$\boxed{\partial_t T + u_\alpha \partial_\alpha T = -\frac{2}{3nk_B}\partial_\alpha h_\alpha - \frac{2}{3nk_B} P_{\alpha\beta} u_{\alpha\beta}}$$

Closure: To compute the evolution of $n, T, \vec{u}$, we need to compute $P_{\alpha\beta} = nM\langle \delta v_\alpha \delta v_\beta\rangle$ & $h_\alpha = \frac{nM}{2}\langle \delta v_\alpha \delta v_\beta \delta v_\beta\rangle$, which we can do perturbatively using f:

$$f_0 \Rightarrow P^0_{\alpha\beta}, h^0_\alpha \Rightarrow \partial_t (n,T,\vec{u})^0 \Rightarrow f_1 \Rightarrow P^1_{\alpha\beta}, h^1_\alpha \Rightarrow \partial_t (n,T,\vec{u})^1 \text{ etc.}$$

Notation: A_0 or A^0 refers to the order of the approximation

2.4.3) Leading order dynamics

Given $n, T, \vec{u}$, we can determine f_0 through:

$$\left.\begin{aligned} &\int f_0 dp = n_0(q) \\ &\int \vec{v} f_0 dp = n_0(\vec{q})\,\vec{u}_0(q) \\ &\int \frac{M}{2}(\vec{v}-\vec{u})^2 f_0 dp = n_0(\vec{q})\,\varepsilon_0(\vec{q}) = \frac{3}{2}k_B T_0(\vec{q}) \end{aligned}\right\} \quad f_0(\vec{q},\vec{p},t) = \frac{n_0(\vec{q})}{[2\pi M k_B T_0]^{3/2}}\, e^{-\frac{(\vec{p}-M\vec{u}_0)^2}{2Mk T_0}} = \frac{n_0}{(2\pi M k_B T_0)^{3/2}}\, e^{-\frac{M\,\delta\vec{v}^2}{2kT_0}}$$

Pressure & heat flux

$$P^0_{\alpha\beta} = n_0 M\langle (v_\alpha - u^0_\alpha)(v_\beta - u^0_\beta)\rangle_0 = n_0 M\,.\,\delta_{\alpha\beta}\,.\,\frac{k_B T_0}{M} = n_0 k_B T_0\,\delta_{\alpha\beta}$$

$\Rightarrow$ Isotropic pressure given by ideal gas law